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Perfect-Fluid Sources for the Levi-Civita Metric II

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Abstract A regular static interior solution of Einstein's field equations representing a perfect fluid cylinder of finite radius is presented. The solution is matched to the Levi-Civita vacuum solution at a boundary where the pressure vanishes. The density and pressure are finite and positive inside the cylinder for a specific range of the mass parameter. The solution could thus represent a reasonable source for the Levi-Civita metric.

Keywords Classical general relativity · Perfect fluid · Static sources

1 Introduction

Cylindrical symmetry means axial symmetry about an infinite axes and translational symmetry along that axes. Then there exist two commuting space like vectors which generate an Abelian group G_2. The reflectional symmetries about and along the axes imply that the two Killing vectors are hypersurface-orthogonal. Static symmetry adds a third Killing vector that is timelike The Levi-Civita metric [1] is the general solution of Einstein's field equations for static cylindrically vacuum. Although the Levi-Civita exterior metric was derived in 1919, the interior field equations were not investigated until 1958 when Marder [2] obtained a solution representing an anisotropic fluid. Other solutions representing clusters of rotating particles have been given [3–6]. The problem for static perfect fluid interior was first solved in 1977 by Evans [7] and quite a few number of solutions, some of them overlapping, have been published since then [8–13]. A class of four static cylindrically symmetric perfect fluid solutions was derived by S. Haggag and Desokey [14]. Also solutions of Kramer's equations for perfect fluid cylinders were obtained [15]. In 2001 a nearly perfect-fluid source was presented [16]. A global solutions of a static perfect fluid cylinders are studied both analytically and numerically [17]. A static cylindrical shell composed of massive particles arising from

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matching of two different Levi-Civita space-times were studied [18]. Also in 2007, solutions in series form were derived [19].

This paper presents regular static interior solution of Einstein's field equations representing a perfect fluid cylinder of finite radius. In Sect. 2 the exterior solution is discussed. A static interior solution is presented in Sect. 3. In Sect. 4 the matching conditions are calculated. Section 5 is devoted to the discussions.

2 The Exterior Solution

The vacuum Levi-Civita metric may be put in the form

$$ds^2 = \rho^{4\sigma} dt^2 - \rho^{8\sigma^2 - 4\sigma}(d\rho^2 + dz^2) - D^2 \rho^{2-4\sigma} d\phi^2 \tag{1}$$

where σ and D are arbitrary constants.

In order to give a geometrical meaning to the radial coordinate ρ we transform it into the proper radius r by defining

$$\rho^{2\sigma(2\sigma-1)} d\rho = dr$$

Thus, we obtain

$$\rho = R^{1/\Sigma}, \qquad R = \Sigma r, \qquad \Sigma \equiv 4\sigma^2 - 2\sigma + 1 \tag{2}$$

With (2) the metric (1) becomes

$$ds^2 = R^{4\sigma/\Sigma} dt^2 - dr^2 - R^{4\sigma(2\sigma-1)/\Sigma} dz^2 - D^2 R^{2(1-2\sigma)/\Sigma} d\phi^2 \tag{3}$$

3 The Interior Solution

Using the proper radius r the static, cylindrically symmetric metric can be written in the general form

$$ds^2 = e^{\nu(r)} dt^2 - dr^2 - e^{\mu(r)} dz^2 - e^{\lambda(r)} d\phi^2 \tag{4}$$

For a perfect fluid the stress-energy tensor has the form

$$T_{ab} = (\varrho + P)\, \dot{x}_a \dot{x}_b - P g_{ab} \quad (a, b = 0, 1, 2, 3)$$

where the four-vector velocity $\dot{x}_a$ is given by $e^{\nu/2}\delta_a^0$.

Einstein's field equations then become

$$-8\pi\varrho = \frac{\lambda'^2}{4} + \frac{\lambda'\nu'}{4} + \frac{\mu'^2}{4} + \frac{\lambda''}{2} + \frac{\mu''}{2} \tag{5}$$

$$8\pi P = \frac{\lambda'\mu'}{4} + \frac{\lambda'\nu'}{4} + \frac{\mu'\nu'}{4} \tag{6}$$

$$8\pi P = \frac{\lambda'^2}{4} + \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\lambda''}{2} + \frac{\nu''}{2} \tag{7}$$

$$8\pi P = \frac{\mu'^2}{4} + \frac{\mu'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\mu''}{2} + \frac{\nu''}{2} \tag{8}$$

where the prime denotes the derivative relative to the radial parameter.

The field equations constitute a system of four equations in five functions. In order to obtain a new solution we put the metric function ν in the form

$$\nu = 4\sigma \left[\frac{r}{d} - (1 - \log d)\right] \tag{9}$$

where d is the proper radius of the source.

From (6–8) we obtain the metric functions λ and μ in the form

$$\lambda = -\frac{c}{a} e^{ar/d} + 2a\frac{r}{d} + h \tag{10}$$

$$\mu = -\frac{c}{a} e^{ar/d} + 2a\frac{r}{d} + k \tag{11}$$

where a, c, h, k are constants

The density and pressure of the fluid are given by

$$\varrho = \frac{1}{16\pi d^2}\left(c\, e^{2\sigma r/d} - 2\sigma\right)\left(12\sigma - c\, e^{2\sigma r/d}\right) \tag{12}$$

$$P = \frac{1}{32\pi d^2}\left(4\sigma - c\, e^{2\sigma r/d}\right)\left(12\sigma - c\, e^{2\sigma r/d}\right) \tag{13}$$

4 Matching Conditions

By matching the Levi-Civita metric (3) to the interior solution (4, 9–11) at the boundary surface $r = d$ we obtain

$$\begin{aligned} a &= 2\sigma \\ c &= 4\sigma e^{-2\sigma} \\ h &= \log \frac{(2-4\sigma)(2-e^{2-4\sigma})}{(e^{2-4\sigma}-1)} \\ k &= \log\left[\Sigma^2 (2 - e^{2-4\sigma})\right] \\ D^2 &= \Sigma^{1/\sigma} \frac{(2-4\sigma)}{(e^{2-4\sigma}-1)} \end{aligned}$$

The density and pressure (12, 13) reduce to the form

$$\varrho = \frac{\sigma^2}{2\pi d^2}\left[2e^{-2\sigma(1-r/d)} - 1\right]\left[3 - e^{-2\sigma(1-r/d)}\right] \tag{14}$$

$$P = \frac{\sigma^2}{2\pi d^2}\left[1 - e^{-2\sigma(1-r/d)}\right]\left[3 - e^{-2\sigma(1-r/d)}\right] \tag{15}$$

It is seen that the pressure vanishes at the boundary surface $r = d$.

Restricting our attention to the inner region, and from (14) and (15) we obtain

$$(\varrho + 2P)^2 = L\,(2\varrho + 5P) \tag{16}$$

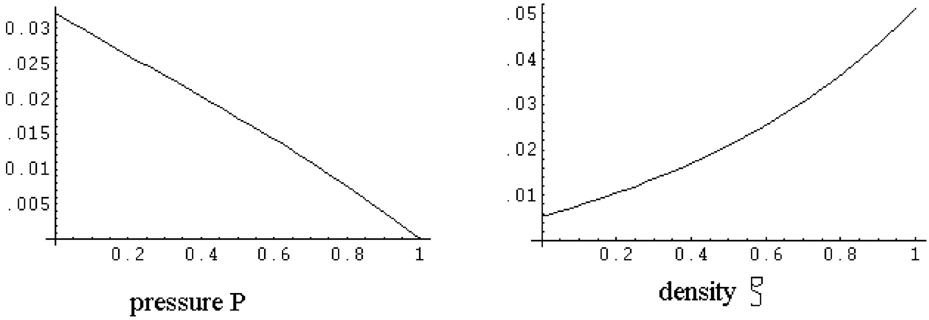

Fig. 1 The pressure and density for the source

where

$$L = \frac{\sigma^2}{2\pi d^2} \tag{17}$$

which gives the equation of state for the perfect fluid.

For positive density and pressure, and for real constants h and k inside the source $r < d$ the mass parameter σ must satisfy

$$\frac{1}{2}\left(1 - \log\sqrt{2}\right) < \sigma < \log\sqrt{2}$$

Thus, the source is valid in the interval

$$0.326713 < \sigma < 0.346574$$

For $\sigma = 0.33$ the plots for the pressure and density are shown in Fig. 1.

A feature of the solution is that the density increases outwards and the pressure decrease outwards. This should be expected as the gradient

$$\varrho' = \frac{\sigma^3}{\pi d^3} e^{-2\sigma(1-\frac{r}{d})}\left[7 - 4e^{-2\sigma(1-\frac{r}{d})}\right]$$
$$P' = \frac{2\sigma^3}{d^3\pi} e^{-4\sigma(1-\frac{r}{d})}\left[1 - 2e^{2\sigma(1-\frac{r}{d})}\right]$$

which is positive in the valid range of σ for the density and negative in the valid range of σ for the pressure. Besides, for the least possible value of σ ($\sigma = 0.326713$) the pressure is less than the density at $r > 0.3794d = r_1$ but is greater than the density at $r < r_1$ and is equal to it at $r = r_1$. Also for the greatest possible value of σ ($\sigma = 0.346574$), the pressure is less than the density at $r > 0.415d = r_2$ but is greater than the density at $r < r_2$ and is equal to it at $r = r_2$.

Then for any possible range of σ, the pressure is less than the density for $r > 0.415d$ i.e. the energy conditions are satisfied for all possible value of σ for a cylindrical shell source $0.415d < r \leq d$.

5 Conclusion

The results obtained above show that the spacetime is regular everywhere. The source is matched to the vacuum Levi-Civita spacetime at the boundary surface $r = d$. The density and pressure are positive inside the source for a specific range of σ and also the pressure is monotonically decreasing outwards. Besides, the energy conditions are satisfied for all possible value of σ for a cylindrical shell source $0.415d < r \leq d$. The solution could thus represents a reasonable perfect-fluid source for the Levi-Civita metric.

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